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Integration of Einstein's Equations in the Weak-Field Domain Using the "Einstein" Gauge

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We propose a new alternative gauge for the Einstein equations instead of the de Donder gauge, which allows in the limit of weak fields a straightforward integration of these equations. The Newtonian potential plays a new and interesting role in this framework. The calculations are demonstrated explicitly for two simple astrophysical models.

1. INTRODUCTION

The usual way to solve the Einstein equations after linearization is first to choose the de Donder or harmonic gauge and second to integrate, in strict analogy to the inhomogeneous Maxwell equations of electrodynamics, via the use of retarded (in order to preserve causality) Green functions.

The common opinion has been that this would be the only generaly way to derive a comprehensive solution of the Einstein equations. This had led to many speculations about the uniqueness of the de Donder gauge, which many scientists have tried to interpret not only as some random mathematical structure, but as something more fundamental—the "physical gauge" (Fock, 1964, §§92, 93). Further fundamental physical arguments for such an opinion have been given.

Nevertheless, in working with rather unconventional methods on finding the correct description for the microscopic gravitational interaction between elementary particles in the realm of quantum physics (Dehnen and Hitzer, 1994, 1995) we were led in a natural way to a new way of integrating the

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Einstein equations in their linearized version. This new method relies on basically two pillars: the use of a special gauge, which Einstein himself had suggested (Einstein, 1916), in order to give his theory an as elegant and concise a shape as possible; and a supplementary gauge condition adding some further specification to Einstein's original suggestion. After implementing this supplemented special gauge, the integration of the Einstein equations after linearization becomes straightforward. So far as we know, such an integration does not appear in the literature.

We have obtained an alternative to the widely accepted standard of the de Donder gauge, so that the arguments about its fundamental physical nature should be carefully reviewed, especially since with the use of the new gauge Einstein's nonlinear theory becomes polynomial. This will be discussed in detail in a future paper.

Yet another highlight concerns the role of Newton's scalar potential, which naturally appears in the Einstein equations in this gauge. No laborious and subtle procedure has to be applied to regain Newton's scalar theory, but its natural embedding easily unfolds.

2. THE EINSTEIN EQUATIONS IN THE SUPPLEMENTED EINSTEIN GAUGE

The general non-Euclidean metric diag $g_{\mu\nu} = (+, -, -, -)$ may be decomposed as

$$g_{\mu\nu} = \eta_{\mu\nu} + \gamma_{\mu\nu} \tag{2.1}$$

where $\eta_{\mu\nu}$ is the usual Minkowski metric (exclusively used for raising and lowering indices) and $|\gamma_{\mu\nu}| << 1$. In the Einstein gauge we have

$$\det(g_{\mu\nu}) \overset{!}{=} -1 \tag{2.2}$$

Inserting (2.1) into (2.2) yields (index raising and lowering only with $\eta_{\mu\nu}$)

$$\gamma := \gamma_{\mu\nu}\eta^{\mu\nu} = 0. \tag{2.3}$$

Using (2.3), the Einstein equations linearized in $\gamma_{\mu\nu}$ can be simplified to

$$\partial^\alpha\partial_\alpha\gamma^{\mu\nu} - \partial_\alpha\partial^\mu\gamma^{\alpha\nu} - \partial_\alpha\partial^\nu\gamma^{\alpha\mu} + \partial_\alpha\partial_\beta\gamma^{\alpha\beta}\eta^{\mu\nu} = -16\pi G T^{\mu\nu} \tag{2.4}$$

We first show in which way the Einstein gauge can be achieved initially assuming $\gamma \neq 0$. The general gauge transformations are

$$x^\mu = x'^\mu + \xi^\mu \tag{2.5a}$$

resulting infinitesimally in

$$\gamma'_{\mu\nu} = \gamma_{\mu\nu} + \partial_\nu\xi_\mu + \partial_\mu\xi_\nu \tag{2.5b}$$

and

$$\gamma' = \gamma + 2\partial_\alpha \xi^\alpha \tag{2.5c}$$

Condition (2.3) of the vanishing of γ' may now be achieved by demanding

$$\partial_\alpha \xi^\alpha \overset{!}{=} -\tfrac{1}{2}\gamma \tag{2.6}$$

But that is only one condition and we will prove that (2.6) may be supplemented by

$$\partial_\nu \gamma'^{\nu}_{\mu} = \partial_\nu \gamma_\mu{}^\nu + \partial_\nu \partial^\nu \xi_\mu + \partial_\mu \partial_\nu \xi^\nu \overset{!}{=} 2\partial_\mu f \tag{2.7}$$

where f is a scalar function of the coordinates. Inserting (2.6) into (2.7) yields

$$\partial_\nu \partial^\nu \xi_\mu - 2\partial_\mu f = \tfrac{1}{2}\partial_\mu \gamma - \partial_\nu \gamma_\mu{}^\nu \tag{2.7a}$$

Equations (2.6) and (2.7a) have the common feature that the unknowns entities ξ^μ and f are on the left-hand sides and the known ones on the right-hand sides. Their solution works as follows. Forming the divergence of (2.7a) yields

$$\partial_\nu \partial^\nu \partial_\mu \xi^\mu - 2\partial_\mu \partial^\mu f = \tfrac{1}{2}\partial_\mu \partial^\mu \gamma - \partial_\mu \partial_\nu \gamma^{\mu\nu} \tag{2.8}$$

Inserting (2.6) a second time and rearranging the terms leads to the determination equation for f:

$$\partial_\mu \partial^\mu f = \tfrac{1}{2}(\partial_\mu \partial_\nu \gamma^{\mu\nu} - \partial_\mu \partial^\mu \gamma) \tag{2.9}$$

Equation (2.9) allows us immediately to calculate f via Green functions. The solution for f may now be inserted into (2.7a) and the vector ξ^μ therefore can be explicitly calculated from

$$\partial_\nu \partial^\nu \xi_\mu = 2\partial_\mu f + \tfrac{1}{2}\partial_\mu \gamma - \partial_\nu \gamma_\mu{}^\nu \tag{2.10}$$

via Green functions. The solutions for f and ξ^μ will obviously satisfy (2.6) and (2.7) and therefore allow us to work in a coordinate system in which in addition to the Einstein gauge (2.2) or (2.3) equation (2.7) holds. Having achieved this, we may rewrite the Einstein equations (2.4) using (2.3) and (2.7) as

$$\partial_\alpha \partial^\alpha \gamma^{\mu\nu} - 4\partial^\mu \partial^\nu f + 2\partial_\alpha \partial^\alpha f \eta^{\mu\nu} = -16\pi G T^{\mu\nu} \tag{2.11a}$$

and the trace equation

$$\partial_\alpha \partial^\alpha f = -4\pi G T \tag{2.11b}$$

(T being the trace of $T^{\mu\nu}$). Because of $\partial_\nu T_\mu{}^\nu = 0$ the conditions (2.3) and (2.7) are in turn consequences of the field equations (2.11) as is the case in the de Donder gauge. Thus the conservation law guarantees the existence of the retarded or advanced integrals of (2.9) and (2.10).

3. THE NEW INTEGRATION PROCEDURE

The general way to solve Einstein's equations (2.11) bearing the special gauge (2.3) in mind will be first to calculate the scalar function f via convolution of the trace of the energy-momentum tensor on the right-hand side of (2.11b) with the well-known retarded Green functions $D(x - x')$ of the D'Alembert operator on the left-hand side:

$$f(x) = 4\pi G \int D(x - x')T(x')\, d^4x' \tag{3.1}$$

The solution for f must then be inserted on the left of (2.11a) in order to obtain

$$\partial_\alpha \partial^\alpha \gamma^{\mu\nu} = -16\pi G\left[T^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}T - \partial^\mu\partial^\nu \int D(x - x')T(x')\, d^4x'\right] \tag{3.2}$$

A second integration via convolution of the right-hand side of (3.2) with the respective Green functions of the D'Alembert operator on the left-hand side results in this final explicit expression for the non-Euclidean deviation $\gamma^{\mu\nu}$ of the general metric from the Minkowski metric:

$$\gamma^{\mu\nu} = 16\pi G \int D(x - x')\Big\{T^{\mu\nu}(x') - \tfrac{1}{2}\eta^{\mu\nu}T(x') - \partial^{\mu'}\partial^{\nu'} \int D(x' - x'')T(x'')\, d^4x''\Big\}\, d^4x' \tag{3.3}$$

It is remarkable that in the vacuum ($T_{\mu\nu} = 0$) equation (3.2) does not go over into a D'Alembert equation, as is the case in the de Donder gauge. Consequently, the general conditions on $T_{\mu\nu}$ for the existence of the integral in (3.3) cannot be given so easily. Instead we discuss two examples in Section 5.

4. THE NEWTONIAN LIMIT

In the Newtonian limit we simply approximate the D'Alembert operator by the three-dimensional space Laplace operator and the trace T of the energy-momentum tensor on the right-hand side of (2.11b) by the matter density ρ:

$$\Delta f = 4\pi G\rho \tag{4.1}$$

which reveals that the scalar function f must be identified with Newton's scalar potential ϕ:

$$f(x) = \phi(x) \tag{4.2}$$

It is therefore obvious that in the specified "Einstein" gauge the trace of the Einstein equations (2.11b) is the general relativistic analogue of the scalar Newtonian equation of gravity. It is shown thereby that a "generalized form" of Newton's scalar potential, the scalar f, is naturally present in the theory of general relativity. We now rewrite the time-time component of equation (3.2) for the static case:

$$\Delta\gamma^{00} = 16\pi G(T^{00} - \tfrac{1}{2}T) \tag{4.3}$$

Replacing T^{00} by the matter density ρ yields the conclusion that γ^{00} is two times f and therefore

$$\gamma^{00}(x) = 2f(x) = 2\phi(x) \tag{4.4}$$

Now, according to the geodesic equation, this γ^{00} suffices to determine the nonrelativistic trajectory of a massive body.

Regarding the swiftness and elegance of this transition from general relativity to Newtonian gravity, it seems to work just as naturally as in the framework of the de Donder gauge. Newton's scalar potential itself achieves new, not only nonrelativistic eminence.

5. SOLUTIONS FOR HOMOGENEOUS AND POLYTROPIC SPHERES

Since gas spheres with homogeneous density ρ may serve as simple models for stars or other astrophysical objects, the solution of Einstein's equations in the presently proposed "Einstein" gauge will be given explicitly. A second, more realistic model, the polytropic gas sphere with zero pressure on its surface, follows.

5.1. Gas Spheres with Homogeneous Density ρ

In the case of static homogeneous density

$$\rho(\mathbf{x}) = \begin{cases} \rho, & |\mathbf{x}| \le R \\ 0, & |\mathbf{x}| > R \end{cases} \tag{5.1}$$

in first approximation the energy-momentum tensor $T^{\alpha\beta}$ takes the simple form

$$T^{\alpha\beta} = \begin{cases} \rho & \text{for} \quad \alpha = \beta = 0 \\ 0 & \text{for} \quad \alpha, \beta \neq 0 \end{cases} \tag{5.2}$$

with the trace

$$T = T^{\alpha\beta}\eta_{\alpha\beta} = \rho \tag{5.3}$$

According to (4.2) and (4.4), f becomes the usual Newtonian scalar potential and γ^{00} its double value,

$$\gamma_{00}(x) = 2f(x) = \begin{cases} \dfrac{MG}{R}\left(\dfrac{r^2}{R^2} - 3\right), & r < R \\ -2\dfrac{MG}{r}, & r > R \end{cases} \tag{5.4a}$$

where

$$M = \tfrac{4}{3}\pi R^3\rho; \qquad r = |\mathbf{x}| \tag{5.5}$$

The space-time components of $\gamma^{\alpha\beta}$ are zero,

$$\gamma_{0\mu} = \gamma_{\mu 0} = 0 \qquad \text{for} \quad \mu = 1, 2, 3 \tag{5.4b}$$

The space components are, according to (3.3),

$$\gamma_{\mu\nu}(x) = \delta_{\mu\nu}\left[\frac{1}{3}f(x) + h(x)\right] - 3\frac{x^\mu x^\nu}{r^2}h(x), \qquad \mu, \nu \in 1, 2, 3 \tag{5.4c}$$

where

$$h(x) = \begin{cases} \dfrac{4}{15}MG\dfrac{r^2}{R^3}, & r < R \\ \dfrac{2}{3}MG\dfrac{1}{r} - \dfrac{2}{5}MG\dfrac{R^2}{r^3}, & r > R \end{cases} \tag{5.4d}$$

It is interesting that for $R > 0$ the gravitational potentials (5.4c) contain for $r > R$ not only $1/r$ terms, but also such terms which decrease with $1/r^3$. Equation (5.4a) already reveals that the gravitational redshift turns out as usual.

The solution for $r > R$ agrees with Einstein's original suggestion for the field of a point mass (Einstein, 1916, p. 819)[2]

[2]Einstein omitted the factor 2 in his formulas.

$$\gamma_{\mu\nu\,\text{Einstein}} = \begin{cases} -2\,\dfrac{GM}{r}, & \mu = \nu = 0 \\ -\dfrac{2GM}{r}\,\dfrac{x^\mu x^\nu}{r^2}, & \mu, \nu \in \{1, 2, 3\} \\ 0, & \text{otherwise} \end{cases} \tag{5.6}$$

up to a term

$$a_{\mu\nu} = \begin{cases} \dfrac{2}{5}\,GMR^2\left(-\delta_{\mu\nu}\,\dfrac{1}{r^3} + 3\,\dfrac{x^\mu x^\nu}{r^5}\right), & \mu, \nu \in \{1, 2, 3\} \\ 0, & \text{otherwise} \end{cases} \tag{5.7}$$

which is such that

$$\Delta a_{\mu\nu} = 0 \tag{5.8}$$

and which vanishes in the point mass limit $R = 0$. Following the original procedure Einstein used to calculate the gravitational bending of light via Huygen's principle (Einstein, 1916)[3] applied to the metric (5.6) and (5.7) yields the usual result.

5.2. The Polytropic Gas Sphere

The general polytropic equation of state is

$$p = \alpha\rho^\gamma, \qquad \alpha, \gamma = \text{const}, \qquad \alpha > 0, \quad \gamma \geq 1 \tag{5.9}$$

where p and ρ are the pressure and density of the gas, respectively, and γ is the so-called polytropic index. It is known that in the Newtonian case the Emden equation is exactly solvable for the physically interesting value $\gamma = 2$ (e.g., matter inside Jupiter). Therefore we restrict ourselves in the following to that case, $\gamma = 2$. Then it can be shown (Dehnen and Obregon, 1971) that the conservation laws yield exactly

$$\rho = \frac{1}{\alpha}\left[\left(\frac{\xi_s}{\xi}\right)^{1/2} - 1\right] \tag{5.10}$$

where ξ is the length of the timelike Killing vector ($\|\nu$ means the covariant differentiation w.r.t. x^ν):

$$\xi_{\mu\|\nu} + \xi_{\nu\|\mu} = 0, \qquad \xi^2 := \xi_\mu \xi^\mu > 0 \tag{5.11}$$

[3] Because Einstein missed a factor 2 in the components of his metric [Einstein (1916), formula (70)], he also obtained only one-half the correct value for the light bending angle (Bergmann, 1942, p. 221).

The index s in (5.10) indicates the evaluation at the surface of the sphere $r = R$.

The coordinates can be chosen such that

$$\xi = \sqrt{g_{00}} = \sqrt{1 + \gamma_{00}} \tag{5.12}$$

which yields for ρ the approximation ($|\gamma_{\alpha\beta}| << 1$)

$$\rho = \frac{1}{4\alpha}(\gamma_{00s} - \gamma_{00}) \tag{5.13}$$

The equation of state (5.9) shows for $\gamma = 2$ that the pressure vanishes in first-order approximation. We again have the same structure of $T^{\alpha\beta}$ as we had in (5.2), with the only difference that ρ itself now depends on the metric.

The solution now is

$$\gamma_{00}(x) = 2f(x) = \begin{cases} -2\dfrac{MG}{R}\left(1 + \dfrac{R}{\pi r}\sin\dfrac{\pi r}{R}\right), & r < R \\ -2\dfrac{MG}{r} & r > R \end{cases} \tag{5.14a}$$

(5.4b) is again valid for the space-time components.

The space-space components are again given by (5.4c). Only the function h is now

$$h(x) = \begin{cases} -4\dfrac{MG}{R}\left\{-\left(\dfrac{R}{r\pi}\right)^3 \sin\left(\dfrac{\pi r}{R}\right) + \left(\dfrac{R}{r\pi}\right)^2 \cos\left(\dfrac{\pi r}{R}\right) + \dfrac{2}{3}\left(\dfrac{R}{r\pi}\right)\sin\left(\dfrac{\pi r}{R}\right)\right\}, & r < R \\ -2\dfrac{MG}{r}\left\{-2\left(\dfrac{R}{r\pi}\right)^2 + \dfrac{1}{3}\dfrac{R^2}{r^2} - \dfrac{1}{3}\right\}, & r > R \end{cases} \tag{5.14b}$$

The solution for $r > R$ again agrees with Einstein's suggestion (5.6) up to a term $a_{\mu\nu}$ which is identical with (5.7) up to a simple change in the numerical factor:

$$\frac{2}{5} \to 2\left(\frac{1}{3} - \frac{2}{\pi^2}\right) \tag{5.15}$$

The remarks below equations (5.4), resp. (5.8), also apply for the gravitational metric of the polytropic gas sphere calculated above. It may be of interest that for other (finite) values of the polytropic index no analytic solution can be given.

5.3. Equivalence with the Schwarzschild Metric

In order to prove in both cases the equivalence of the calculated metrics for $r > R$ outside the spheres with the Schwarzschild metric (radial coordinate r_S) one needs to write both metrics in polar coordinates. Rescaling the radial coordinate by

$$r_S^2 = r^2\left(1 + FGM\frac{R^2}{r^3}\right) \tag{5.16a}$$

with

$$F = \begin{cases} \dfrac{2}{5} & \text{for the homogeneous gas sphere} \\ 2\left(\dfrac{1}{3} - \dfrac{2}{\pi}\right) & \text{for the polytropic gas sphere} \end{cases} \tag{5.16b}$$

yields the usual form of the Schwarzschild metric in first-order approximation.

6. CONCLUSIONS

The proposed new gauge and the new way for solving the Einstein equations may seem in the weak-field domain only as some kind of technical alternative apart from the new light they shed on the role of the Newtonian potential even in the theory of general relativity. But the general nonlinear form of the special "Einstein" gauge is under investigation. The results will be presented in a future paper.

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